Math 434 Assignment 4

Due May 10

Assignments will be collected in class.

4. Let T be a complete theory. A *complete n-type (of T)* $p(x_1, \dots, x_n)$ is a set of formulas in n free variables $x_1, \dots, x_n$ such that:

   - for every formula $\psi(x_1, \dots, x_n) \in p$, $T \vDash \exists x_1, \dots, x_n \psi(x_1, \dots, x_n)$;
   - for every formula $\psi(x_1, \dots, x_n)$, either $\psi \in p$ or $\neg\psi \in p$.

   We say that a model $\mathcal{M}$ realizes a complete n-type p if there are $a_1, \dots, a_n \in \mathcal{M}$ such that for every $\psi \in p$, $\mathcal{M} \vDash \psi(a_1, \dots, a_n)$.

   Show that there is a model of T that realizes every complete n-type of T.

   Solution: We suppose that the language is countable, so that there are only countably many formulas. Let $\mathcal{M}$ be a model of T and let $\mathcal{U}$ be a non-principal ultrafilter on ω. Let $\mathcal{A} = \prod \mathcal{M}/\mathcal{U}$.

   Fix an arbitrary n-type p. Let $\varphi_1, \varphi_2, \dots$ be a list of the formulas in p. For each i, there are $a_1^i, \dots, a_n^i \in \mathcal{M}$ such that

   $$\mathcal{M} \vDash \varphi_1(a_1^i, \dots, a_n^i) \wedge \cdots \wedge \varphi_i(a_1^i, \dots, a_n^i).$$

   Let $a_j = [a_1^i] \in \mathcal{A}$. We claim that $a_1, \dots, a_n$ realizes p in $\mathcal{A}$. Indeed, for each formula $\varphi_i \in p$,

   $$\{j \in \omega : \mathcal{M} \vDash \varphi_i(a_1^j, \dots, a_n^j)\} \supseteq \{j \in \omega : j \geq i\} \in \mathcal{U}$$

   and so by Los's theorem, $\mathcal{A} \vDash \varphi_i(a_1, \dots, a_n)$.

7. Except for the axiom of infinity, all the axioms of ZFC hold in V_ω. Prove that the axiom of infinity does not hold, i.e., $V_\omega \vDash \neg(\exists x)[\varnothing \in x \wedge (\forall y \in x)(\{y\} \cup y \in x)]$. (This shows that the axiom of infinity is not implied by the remaining axioms of ZFC.)

   Solution: Suppose that there is $x \in V_\omega$ such that V_ω thinks that x is inductive. Being inductive is Δ_0 so x is actually inductive. Since ω is the smallest inductive set, $\omega \subseteq x$. There is $n < \omega$ such that $x \in V_n$, so that $\omega \subseteq x \subseteq V_n$; then $\omega \in V_{n+1}$. But this cannot be true, a contradiction.